

LOWER CONSISTENCY BOUNDS FOR MUTUAL STATIONARITY WITH DIVERGENT UNCOUNTABLE COFINALITIES

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ABSTRACT. We prove that the upper bounds for the consistency strength of certain instances of mutual stationarity considered by Liu-Shelah [8] are close to optimal. We also consider some related and, as it turns out, stronger properties.

1. INTRODUCTION

Mutual stationarity was originally introduced in [3] to study saturation properties of non-stationary ideals.

Definition 1. *Let λ be an ordinal, any ordinal. Let $\langle \kappa_i : i < \lambda \rangle$ be an increasing sequence of regular cardinals, $\bar{\kappa} := \sup_{i < \lambda} \kappa_i$. We say a sequence $\langle S_i : i < \lambda \rangle$, where $S_i \subseteq \kappa_i$ is stationary, is mutually stationary iff the set*

$$\{A \subset \bar{\kappa} \mid \forall i < \lambda : \kappa_i \in A \Rightarrow \sup(A \cap \kappa_i) \in S_i\}$$

is stationary, i.e. contains a substructure of every structure with countable signature on $\bar{\kappa}$.

The most remarkable result from the above paper is the ZFC fact that any sequence of stationary sets all of which concentrate on points of countable cofinalities is mutually stationary, no matter its length. It is also shown that an analog theorem for sets concentrating on cofinality ω_1 can not be proven in ZFC. We do not currently know if it is even consistent, though a lower bound for its consistency is known (see [7],[10]).

However, we are only going to discuss sequences that do not concentrate on a fixed cofinality. We shall also limit ourselves to stationary subsets of the \aleph_n 's, n a natural number.

All mutually stationary sequences appearing in the paper will have limit length.

We will mention some prior results involving sets concentrating on countable cofinality to draw some parallels with the results from this paper. We start with this result:

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Theorem 2 (Magidor). *Let $\langle \kappa_i : i < \omega \rangle$ be an increasing sequence of measurable cardinals. Then there exists a generic extension of the universe $V[G]$ in which κ_i becomes \aleph_{2i+1} and the sequence $\langle S_0^2, S_1^3, S_0^4, S_1^5, S_0^6, \dots \rangle$ is mutually stationary.*

(Note: We will often use the following notation: $S_n^m := \{\alpha < \aleph_m \mid \text{cof}(\alpha) = \aleph_n\}$.)

The theorem can be stated more generally, the real limitations being that all but finitely many sets in the sequence concentrate on one of two cofinalities, ω being one of them, and if a set in the sequence concentrates on countable cofinality then the next one does not. Here the points in the sequence concentrating on countable cofinality correspond to former measurable cardinals and their successors are not collapsed in the construction and correspond to points concentrating on the other cofinality.

If one wants to do away with this non-accumulation property of points concentrating on countable cofinality one uses supercompact cardinals instead¹. In that case for any given $f : \omega \rightarrow 2$ in the ground model there exists a generic extension in which the sequence $\tilde{S}_2^f := \langle S_{f(m)}^m : 2 \leq m < \omega \rangle$ is mutually stationary (see [2]).

This result can be improved using a completely different approach. Jensen has shown the consistency of a forcing axiom (relative to one supercompact) that implies the mutual stationarity of \tilde{S}_2^f for all $f : \omega \rightarrow 2$ simultaneously. (See [5].)

The Magidor result, too, can be improved:

Theorem 3 (Koepeke). *Let κ be a measurable cardinal. Then there exists a generic extension of the universe $V[G]$ in which κ becomes \aleph_ω and the alternating sequence $\langle S_0^2, S_1^3, S_0^4, S_1^5, S_0^6, \dots \rangle$ is mutually stationary.*

It is not hard to see that this is optimal. There is an interesting switch that happened here. In the Koepeke result different limitations apply: all but finitely many sets in the sequence concentrate on one of two cofinalities, ω being one of them, and if a set in the sequence concentrates on the *other* cofinality then the next one does not. Here the points in the sequence concentrating on the *other* cofinality correspond to points in a Prikry sequence and their successors are not collapsed in the construction and correspond to points concentrating on *countable* cofinality. (See [6].)

This leads us to ask the following question. Is it possible to force the mutual stationarity of the sequence $\langle S_0^2, S_1^3, S_1^4, S_0^5, S_1^6, S_1^7, \dots \rangle$ from finitely many measurable cardinals?

From now on, all sets in a mutually stationary sequence will concentrate on points of uncountable cofinality. The following result is an analog to Magidor's result above.

Theorem 4 (Liu-Shelah). *Let $1 \leq m < k$ be natural numbers. Let $A \subset \omega$ be infinite s.t.*

$$n \in A \Rightarrow n + 1 \notin A$$

¹Successors of supercompacts might be collapsed in this construction, this being a classic use of indestructibility.

for all $n < \omega$. Let $f : \omega \rightarrow \{n, k\}$ be defined by

$$f(n) := \begin{cases} m & n \in A \\ k & n \notin A. \end{cases}$$

Let $\langle \kappa_i : i < \omega \rangle$ be an increasing sequence of cardinals of Mitchell order at least $\omega_m + 1$. Then there exists a generic extension in which $\langle \kappa_i : i < \omega \rangle$ is the increasing enumeration of $\langle \aleph_n : n > k, n \in A \rangle$ and the sequence $\vec{S}_{k+1}^f := \langle S_{f(n)}^n : k < n < \omega \rangle$ is mutually stationary.

(There was a significantly weaker precursor result in [4], but it has been superseded by this one from [8].)

We do not know about a higher level analog to the Koepke result, but we think that it should exist.

The Liu-Shelah paper [8] has another result, one which is nominally very powerful.

Theorem 5 (Liu-Shelah). *Assume $\max(\text{pcf}(\{\aleph_n : n < \omega\})) = \aleph_{\omega+n^*}$. Let $1 < m^* < \omega$. Let I be the ideal of finite subsets of ω . Let $\langle A_i : i < n^* \rangle$ be a partition of ω such that $\prod_{k \in A_i} \aleph_k / I$ has true cofinality $\aleph_{\omega+i+1}$ for $i < n^*$. Let $\langle n_i : i < n^* \rangle \subset [1, m^*]$ be arbitrary. Define a function $f : \omega \rightarrow \omega$ by*

$$f(n) = n_i :\Leftrightarrow n \in A_i.$$

Then the sequence $\vec{S}_{m^+1}^f$ is mutually stationary.*

Note that the requirement here is the failure of SCH at \aleph_ω . So, we are still below 0^\sharp . We are interested to know if this theorem can be used to generate mutually stationary sequences not already covered by Theorem 4. For that end we do need the ability to control for the partition $\langle A_i : i < n^* \rangle$ ². Unfortunately, we do not know how to do that. (See also Question 34.)

We now state the main results of this paper. Theorem 6 shows that the upper bounds obtained by Liu-Shelah in Theorem 4 are close to optimal:

Theorem 6. *Let $1 < m$ be a natural number. Suppose $\langle S_n \mid n \geq m+1 \rangle$ is a mutually stationary sequence such that:*

- (1) *for every $n \geq m+1$, S_n is stationary in ω_n and concentrates on a fixed cofinality μ_n ;*
- (2) *$\langle \mu_n \mid n \geq m+1 \rangle$ is **not** eventually constant; and*
- (3) *$\omega_1 \leq \mu^* := \liminf_{n \rightarrow \infty} \mu_n < \aleph_\omega$.*

Then there is an inner model W such that: for infinitely many $n \in \omega$:

$$V \models \{ \alpha < \omega_n \mid o^W(\alpha) \geq \mu^* \} \text{ is stationary in } \omega_n$$

The hypotheses of Theorem 6 are consistent, by the Liu-Shelah Theorem 4. For example, mutual stationarity of the sequence

$$\langle S_2^3, S_1^4, S_2^5, S_1^6, \dots, S_2^{2k-1}, S_1^{2k}, \dots \rangle$$

falls under the hypothesis of Theorem 6 (with $\liminf_{n \rightarrow \infty} \mu_n = \omega_1$).

²Doing so might necessitate large cardinals beyond 0^\sharp .

We do not know if the hypotheses of these following theorems is consistent. Theorem 7 and Theorem 8 have analogs in the countable case, mentioned in the introduction, which we do know to be consistent. Therefore we are confident that these hypotheses will be found to be consistent in the end. We are less confident about Theorem 9, but will include it anyway as it presents only a minimal time investment.

Furthermore, these hypotheses cover the most obvious variations of the hypothesis of our main theorem, Theorem 6. We feel the paper would be incomplete without addressing them.

Theorem 7. *Assume 0^\sharp does not exist. Fix natural numbers $l > 1, m > 1$. Suppose $\langle S_n \mid n \geq m+1 \rangle$ is a sequence such that for every $n \geq m+1$:*

- (1) *S_n is stationary in ω_n and concentrates on a fixed uncountable cofinality μ_n ; and there exists a strictly increasing sequence $\langle n_k : k < \omega \rangle$ with*
- (2) *$n_{k+1} \geq n_k + l$ for all $k < \omega$*
- (3) *$\langle \mu_{n_k} \mid k < \omega \rangle$ is not eventually constant*
- (4) *$\mu_{n_k} = \mu_{n_k+i}$ for all $k < \omega$ and $i < l$*
- (5) *$\langle S_n \mid n \geq m+1 \rangle$ is mutually stationary.*

Then in K there is an infinite sequence $\langle \kappa_n : n < \omega \rangle \subset \{\aleph_n : n < \omega\}$ s.t for all $n < \omega$ there is $\kappa < \kappa_n$ s.t $(\kappa^+)^K < \kappa_n$ and $\text{o}^K(\kappa) \geq (\kappa_n)^{+(l-1)}$.

Theorem 8. *Let $1 \leq n, k < m < \omega$ and assume that the sequence $\langle S_{f(n)}^{n+m} : n < \omega \rangle$ is mutually stationary for all $f : \omega \rightarrow \{n, k\}$. Then 0^\sharp exists.*

By the results of Liu-Shelah mentioned in Theorem 4, our Theorem 6 is almost an equiconsistency. However, if we alter the assumption of Theorem 6 to require that $\liminf_{n \rightarrow \infty} \mu_n = \aleph_\omega$, the consistency strength jumps considerably, as shown by the following Theorem 9. In fact, the hypotheses of Theorem 9 is an apparent strengthening of stating that \aleph_ω is a Jonsson cardinal, which is not known to be consistent at all.

Theorem 9. *Fix $1 \leq m < \omega$. Suppose $\langle S_n \mid n \geq m \rangle$ is a mutually stationary sequence such that for every $n \geq m$:*

- (1) *S_n is stationary in ω_n and concentrates on a fixed cofinality μ_n ;*
- (2) *$\liminf_{n \rightarrow \infty} \mu_n = \aleph_\omega$.³*

Then 0^\sharp exists.

2. PRELIMINARIES

2.1. Inner model theory. Unless otherwise stated, we follow the conventions of Zeman [11], assume that 0^\sharp does not exist, and let K denote the core model (see Chapter 8 of [11]). Like [11], we use Jensen indexing of extenders. We will heavily depend on the following lemma.

Lemma 10. *Let M be a premouse. Let n and κ be such that M is $n+1$ -sound above κ . Assume that $\lambda \in M$ is such that*

$$\kappa < \lambda \leq \rho_n^M$$

³Equivalently, any given cofinality appears only boundedly often in $\langle \mu_n : n \geq m \rangle$.

and $\text{cof}^M(\lambda) > \kappa$. Then $\text{cof}^V(\lambda) = \text{cof}^V(\rho_n^M)$.

Proof. We can assume that $n = 0$, otherwise replace M by its n -th reduct. Define $f : \text{On} \cap M \rightarrow \lambda$ by

$$\xi \mapsto \sup(\text{Hull}_1^{M||\xi}(\kappa \cup \{p_1^M\}) \cap \lambda).$$

By assumption this is well-defined and cofinal. It is also clearly increasing. Hence, we are done. \square

We will need the following basic fact about normal fine-structural iterations.

Fact 11 (See Lemma 4.2.2 of [11]). *Suppose $\langle M_i \mid i \leq \theta \rangle$ is a normal fine-structural iteration of a premouse $M = M_0$. Let κ_i denote the critical point of the i -th stage. Assume that the ultimate projectum of M_0 is $\leq \kappa_0$. Then for every $i < \theta$, the ultimate projectum of M_i is $\leq \kappa_i$. Let $\text{deg}(M_i, \kappa_i)$ denote the maximal $n \in \omega$ such that $\kappa_i < \omega \rho_n^M$. If θ is a limit ordinal, then $\langle \text{deg}(M_i, \kappa_i) \mid i < \theta \rangle$ is eventually constant.*

2.2. Facts about mutual stationarity. The following lemma will be used to modify the members of sets witnessing mutual stationarity:

Lemma 12. *Suppose $\langle S_n \mid n \geq n_0 \rangle$ is a sequence such that S_n is a stationary subset of ω_n for every $n \geq n_0$. Fix an algebra $\mathfrak{A} = (H_{\aleph_{\omega+1}}, \in, \dots)$ and assume that $X \prec \mathfrak{A}$ and $\sup(X \cap \omega_n) \in S_n$ for every $n \geq n_0$. Fix a regular uncountable $\mu < \aleph_\omega$ and set*

$$X' := Sk^{\mathfrak{A}}(X \cup \mu)$$

Then for all n such that $\omega_n > \mu$:

$$\sup(X' \cap \omega_n) = \sup(X \cap \omega_n)$$

Proof. The \geq direction is trivial. For the \leq direction, fix an n such that $\mu < \omega_n$. Let η be an element of $\omega_n \cap X'$. Then there is a function $f \in X$ and an ordinal $\xi < \mu$ such that $\eta = f(\xi)$. Let h be the restriction of f to those inputs from μ whose outputs are below ω_n . Since μ is among the \aleph_k 's then $\mu \in X$, and so since $f \in X$ it follows that $h \in X$. Since ω_n is regular and $> \mu$ then $\sup(\text{range}(h)) \in X \cap \omega_n$. Thus $\eta = f(\xi) = h(\xi) < \sup(X \cap \omega_n)$. \square

Corollary 13. *Suppose $\vec{S} = \langle S_n \mid n \geq n_0 \rangle$ is mutually stationary, where $S_n \subset \omega_n$ for each $n \geq n_0$. Let $\mu < \aleph_\omega$ be fixed, and let n_1 be such that $\mu < \omega_{n_1}$. Then the mutual stationarity of $\langle S_n \mid n \geq n_1 \rangle$ is witnessed by models which contain μ as a subset.*

The following lemma can be easily proved by induction on n :

Lemma 14. *Assume $\mu < \aleph_\omega$ is regular, $\mu \subset X \prec H_{\aleph_{\omega+1}}$, and $\sup(X \cap \omega_n)$ has cofinality $\geq \mu$ whenever $\omega_n \geq \mu$. Then for every such n , every $< \mu$ -sized subset of $X \cap \omega_n$ is covered by some $< \mu$ -sized set from X . In particular, $X \cap \aleph_\omega$ is a $< \mu$ -closed set of ordinals.*

3. PROOF OF THEOREM 6

In this section we prove Theorem 6. Define

$$\mu^* := \liminf_{n \rightarrow \infty} \mu_n$$

Recall we are assuming that

$$(1) \quad \mu^* < \aleph_\omega$$

Remark 15. *The case where $\mu^* = \aleph_\omega$ is Theorem 9. However, unlike the assumptions of Theorem 6, the assumptions of Theorem 9 are not known to be consistent.*

As described in Section 2, we work with the core model K below 0^\sharp .⁴

First we state a couple of theorems which are proved in [1]:

Theorem 16 ([1], Lemma 44). *Let K be the core model below 0-pistol and λ an uncountable cardinal. Assume S is a stationary collection of $X \prec H_\lambda$ such that*

$$\text{cof}(\omega) \cap \lambda \cap \lim(X \cap \lambda) \subset X$$

For each $X \in S$ let $\sigma_X : H_X \rightarrow X \prec H_\lambda$ be the inverse of the Mostowski collapse of X , and let $K_X := \sigma_X^{-1}[K \cap H_\lambda]$. Then for all but nonstationarily many $X \in S$, in the coiteration of K with K_X :

- *The K side truncates to a mouse of size at most $|\text{crit}(\sigma_X)|$ by stage 1 of the coiteration;*
- *the K_X side of the coiteration is trivial.*

Notation 17. *Let S be as in the hypothesis of Theorem 16. For each $X \in S$ we let θ_X denote the length of the K versus K_X coiteration, and let $\langle N_i^X, \kappa_i^X, E_i^X \mid i < \theta_X \rangle$ denote the sequence of mice, critical points, and applied extenders on the K side of the coiteration.⁵ For $i \leq j < \theta_X$ let $\pi_{i,j}^X$ denote the (possibly partial) iteration map from $N_i^X \rightarrow N_j^X$.*

The following theorem was a generalization of a Covering Theorem of Mitchell:⁶

Theorem 18 (Theorem 1 of Cox [1]). *Assume 0^\sharp does not exist, and let K be the core model. Suppose γ is an ordinal, $\gamma > \omega_2$, $\text{cf}(\gamma) < |\gamma|$, and γ is regular in K . Then γ is measurable in K . Moreover, if $\text{cf}(\gamma) > \omega$ then in K , γ has Mitchell order at least $\text{cf}^V(\gamma)$.*

We now commence with the proof of Theorem 6. Fix a large regular θ and a structure $\mathfrak{A} = (H_\theta, \in, \vec{S}, \dots)$ for the remainder of the proof. For each X witnessing mutual stationarity of \vec{S} , let $\sigma_X : H_X \rightarrow X \prec \mathfrak{A}$ be the inverse of the collapsing map of X and let K_X denote $\sigma_X^{-1}[K \cap H_\theta]$.

Recall that we are assuming $\mu^* = \liminf_{n \rightarrow \infty} \mu_n < \aleph_\omega$. By Corollary 13, if we let m_1 be large enough so that $\omega_{m_1} > \mu^*$, then the mutual stationarity of $\langle S_n \mid n \geq m_1 \rangle$ is witnessed by sets containing μ^* as a subset; let T denote this stationary set.

⁴If 0^\sharp exists then by iterating 0^\sharp one easily obtains an inner model as in the conclusion of Theorem 6.

⁵Recall from Theorem 16 that the K_X side of the coiteration is trivial.

⁶E.g. it removed all cardinal arithmetic assumptions from the hypotheses.

Lemma 14, together with the fact that $\mu_n \geq \mu^*$ for all $n \geq m_1$ and $\mu^* \subset X$ for all $X \in T$, yields:

Observation 19. *For every $X \in T$, $X \cap \aleph_\omega$ is closed under limits of cofinality less than μ^* . In particular, since Theorem 6 assumes that $\mu^* \geq \omega_1$, then $X \cap \aleph_\omega$ is an ω -closed set of ordinals and thus Theorem 16 applies.*

For $X \in T$ let $\beta_\omega^X := \sigma_X^{-1}(\aleph_\omega)$. By Observation 19 and Theorem 16, for every $X \in T$ the following facts hold for the coiteration of K with $K_X || \beta_\omega^X$:

(2) the K versus $K_X || \beta_\omega^X$ coiteration is trivial on the $K_X || \beta_\omega^X$ side

and

(3) K truncates to a mouse of size at most $|\text{crit}(\sigma_X)|$ by stage 1

For each $X \in T$ and $n > m_1$ let

$$\beta_n^X := \sigma_X^{-1}(\omega_n)$$

Since $\text{cf}(X \cap \omega_n) = \mu_n$ for all $n > m_1$, then $\text{cf}^V(\beta_n^X) = \mu_n$. So the assumptions of the theorem imply that for every $X \in T$:

(4) $\langle \text{cf}^V(\beta_n^X) \mid n > m_1 \rangle$ is not eventually constant

Let θ_X denote the length of the coiteration of K with $K_X || \beta_\omega^X$; equivalently, θ_X is the least stage of the K versus K_X coiteration such that all disagreements below β_ω^X have been resolved. Let $\langle N_i^X, \kappa_i^X, \nu_i^X \mid i < \theta_X \rangle$ denote the mice, critical point, and iteration index of the mouse on the K -side of the coiteration of K with $K_X || \beta_\omega^X$. Note that by (2) it follows that for all $i < \theta_X$:

(5) $\nu_i^X = o^{K_X}(\kappa_i^X)$

The following argument is due to Magidor:

Lemma 20 (Magidor [9]). *For every $X \in T$:*

$$\{\kappa_i^X \mid i < \theta_X\} \cap \beta_\omega^X \text{ is cofinal in } \beta_\omega^X$$

Proof. Assume not. By (2) and universality of K , $M_{\theta_X}^X$ end extends $K_X || \beta_\omega^X$. Let η_X be the strict supremum of $\{\kappa_i^X \mid i < \theta_X\}$; by assumption, $\eta_X < \beta_\omega^X$. Now (3) implies that $M_{\theta_X}^X$ projects below η_X and is sound above η_X .⁷ Let M be the maximal initial segment of $M_{\theta_X}^X$ such that β_ω^X is a cardinal in M . If $M = M_{\theta_X}^X$ then we have already shown that there is some $\eta < \beta_\omega^X$ such that M projects below η and is sound above η . If M is a proper initial segment of $M_{\theta_X}^X$ then, since β_ω^X is definably collapsed over M , it follows that M projects strictly below β_ω^X and, being a proper initial segment of a mouse, is (fully) sound. In either case there are $n^*, m^* < \omega$ such that

$$\rho_{n^*+1}^M \leq \beta_k^X < \beta_\omega^X \leq \rho_{n^*}^M$$

for all $k \geq m^*$. Fix any $k > m^*$. Since β_k^X is regular in K_X , β_ω^X is a cardinal in M , and M end-extends $K_X || \beta_\omega^X$, it follows by acceptability that β_k^X is regular in M .

⁷This is a routine inductive proof; see e.g. the proof of Lemma 6.6.4 of Zeman [11].

But then by Lemma 10 together with the soundness properties of M established above, $\text{cof}(\beta_k^X) = \text{cof}(\rho_{n^*}^M)$ for all but finitely many k . This contradicts (4). \square

Note that Lemma 20 implies that:

$$(6) \quad \forall X \in T \quad \theta_X \text{ is a limit ordinal}$$

Lemma 20, together with the fact that there are only finitely many truncations in an iteration, yield that for every $X \in T$ there is an $n_X \in \omega$ such that, whenever $i < \theta_X$ and $\kappa_i^X \geq \beta_{n_X}^X$, then i is not a truncation stage; i.e. all truncations of the K versus $K_X || \beta_\omega^X$ coiteration must occur before the critical points reach $\beta_{n_X}^X$. Using (6) and Fact 11, it follows that for each $X \in T$ the sequence

$$\langle \deg(N_i^X, \kappa_i^X) \mid \kappa_i^X \geq \beta_{n_X}^X \text{ and } i < \theta_X \rangle$$

is, eventually, a constant sequence of natural numbers.

So by increasing n_X if necessary, we may also assume that $\deg(N_i^X, \kappa_i^X)$ is constant with value m_X for all i such that $\kappa_i^X \geq \beta_{n_X}^X$. By countable completeness of the nonstationary ideal:

$$(7) \quad \exists m^*, n^* \in \omega \quad \exists T' \subset T \text{ stationary } \forall X \in T' \quad n_X = n^* \text{ and } m_X = m^*$$

Let $X \in T'$. Since (total) iteration maps are cofinal, we have that the cofinality of $\rho_{m^*}(N_i^X)$ is constant for all i which satisfy:

$$(8) \quad \beta_{n^*}^X \leq \kappa_i^X < \beta_\omega^X$$

For each $X \in T'$ let λ_X denote the constant cofinality of $\rho_{m^*}(N_i^X)$, for those i satisfying (8).

For each $n \in \omega$ define:

$$(9) \quad \theta_n^X := \text{the least stage such that } \kappa_{\theta_n^X}^X \geq \beta_n^X$$

Note that:

$$(10) \quad \forall n \in \omega \quad \left| K_X || \beta_n^X \right| = \left| X \cap \omega_n \right| < \aleph_\omega$$

Combined with (3) and Lemma 4.4.1 of Zeman [11], this implies that $|N_i^X| < \aleph_\omega$ for all $i \in (1, \theta_X)$. In particular:

$$(11) \quad \forall X \in T' \quad \lambda_X < \aleph_\omega$$

So $\lambda_X \in \{\omega_k \mid k \in \omega\} \subset X$. Thus by pressing down there is some fixed infinite cardinal $\lambda^* < \aleph_\omega$ and a stationary $T'' \subset T'$ such that $\lambda_X = \lambda^*$ for all $X \in T''$. Since $\langle \mu_n \mid n > m_1 \rangle$ is not eventually constant:

$$(12) \quad I := \{n \in \omega \mid \lambda^* \neq \mu_n(\text{cf}^V(\beta_n))\} \text{ is infinite}$$

We now consider two cases. If, for some $X \in T''$ and $n \in I \cap (n^*, \omega)$, there is an iterate N_i^X such that $\text{crit}(E_i^X) < \beta_n^X$ but the generators of E_i^X are cofinal in β_n^X , then by iterating this extender we can obtain a model as in the conclusion of Theorem 6. **So from now on we assume there is no such extender**, i.e. assume:

$$(13) \quad \forall X \in T'' \quad \forall n \in I \cap (n^*, \omega) \quad \forall i < \theta_n^X \\ \text{the generators of } E_i^X \text{ are bounded below } \beta_n^X$$

Lemma 21. *For every $X \in T''$ and for all $n \in I \cap (n^*, \omega)$: the critical points of the coiteration are cofinal in β_n^X .*

Proof. Fix $n > n^*$ s.t. $n \in I$; i.e. $\text{cof}(\beta_n^X) \neq \lambda^* = \lambda_X$. Now let us assume for a contradiction that there is an i with $\kappa_i^X < \beta_n^X$ but $\kappa_{i+1}^X \geq \beta_n^X$. Note that β_n^X is regular in K_X . Since $K_X \parallel \beta_\omega^X$ doesn't move in the coiteration, $i > n^*$ (in particular i isn't a truncation stage), and by acceptability, it follows that β_n^X is also regular in N_{i+1}^X . Furthermore, assumption (13) implies that the generators of E_i^X are bounded by some $\zeta < \beta_n^X$, which in turn implies that N_{i+1}^X is sound above $\zeta + 1$. Also $\rho_{m^*}(N_{i+1}^X) \geq \kappa_{i+1}^X \geq \beta_n^X$ (recall m^* was defined in (7) as the uniform eventual value of $\deg(N_j^X, \kappa_j^X)$). So we can apply Lemma 10 to conclude that $\text{cof}(\beta_n^X) = \lambda_X$. But this contradicts our choice of n ! \square

In particular if $X \in T''$ and $n \in I \cap (n^*, \omega)$, then θ_n^X is a limit ordinal and $\text{cf}^V(\theta_n^X) = \text{cf}^V(\beta_n^X) = \mu_n$; here θ_n^X is as defined in (9).

Lemma 22. *Let $X \in T''$ and $n \in I \cap (n^*, \omega)$. Then the following set is closed and unbounded in θ_n^X :*

$$C_n^X := \{j < \theta_n^X \mid \pi_{j, \theta_n^X}^X(\kappa_j^X) = \beta_n^X\}$$

Proof. First we show that C_n^X is unbounded in θ_n^X . Assume not, and let $i_0 < \theta_n^X$ be a bound on C_n^X . By Lemma 21, θ_n^X is a limit ordinal. So there is some $j^* \in (i_0, \theta_n^X)$ such that β_n^X has a preimage in $N_{j^*}^X$, say $\bar{\beta}$. We claim that

$$(14) \quad \kappa_{j^*}^X < \bar{\beta}$$

Suppose not. Our assumptions imply that these two ordinals are not equal, so it must be that $\kappa_{j^*}^X > \bar{\beta}$. But $\kappa_{j^*}^X < \beta_n^X$ (since $j^* < \theta_n^X$), so since $\pi_{j^*, \theta_n^X}^X \upharpoonright \kappa_{j^*}^X = \text{id}$ this would imply that $\beta_n^X < \beta_n^X$, a contradiction.

Since β_n^X is regular in $K_X \parallel \beta_\omega^X$, θ_n^X is past all truncation points of the K versus $K_X \parallel \beta_\omega^X$ coiteration, and K_X does not move in the coiteration, it follows that β_n^X is regular in $N_{\theta_n^X}^X$. So by elementarity of the iteration map:

$$(15) \quad \bar{\beta} \text{ is regular in } N_{j^*}^X$$

So, our iteration embeddings are continuous at $\bar{\beta}$ and thus $\text{cof}(\bar{\beta}) = \text{cof}(\beta_n^X) \neq \lambda_X$, where the latter inequality is because $n \in I$.

On the other hand $N_{j^*}^X$ is sound above $\kappa_{j^*}^X$, $\bar{\beta}$ is regular in $N_{j^*}^X$ by (15), and $\bar{\beta}$ is strictly above $\kappa_{j^*}^X$ by (14). So we can conclude by Lemma 10 that $\text{cof}(\bar{\beta}) = \lambda_X$. This is a contradiction, completing the proof that C_n^X is unbounded. That C_n^X is closed below θ_n^X is a routine exercise, using the fact that the critical points of the iteration are increasing. \square

Let I' denote the tail end of I beyond n^* , and also ensure that

$$(\mu^*)^+ < \omega_{\min(I')}$$

For the rest of the proof, fix some $n \in I'$; by (12) there are infinitely many such n . Also fix some $X \in T''$. Observe that if C_n^X is as in the statement of Lemma 22,

then

$$D_n^X := \{\alpha \mid \exists j \in C_n^X \ \alpha = \kappa_j^X\} \text{ is club in } \beta_n$$

Also observe that if $j \in C_n^X$ then since j is past all truncations, κ_j^X is a regular cardinal in the j -th iterate of K_X ; but since K_X doesn't move in the coiteration this just means κ_j^X is regular in K_X . Thus

$$\forall \alpha \in D_n^X \ K_X \models \alpha \text{ is regular}$$

and so by elementarity of σ_X it follows that:

$$(16) \quad \forall \alpha \in \tilde{D}_n^X := \sigma_X[D_n^X] \ K \models \alpha \text{ is regular}$$

By Observation 19, \tilde{D}_n^X is closed under limits of cofinality $< \mu^*$. Also \tilde{D}_n^X is cofinal in $\sup(\sigma_X[\beta_n]) = \sup(X \cap \omega_n)$. Together with (16) it follows that

$$(17) \quad \begin{aligned} &\forall \eta \in \lim(\tilde{D}_n^X) \cap \text{cof}(\geq \mu^*), \text{ all but nonstationarily} \\ &\text{many members of } \eta \cap \text{cof}(< \mu^*) \text{ are regular in } K \end{aligned}$$

The notation $\lim(\tilde{D}_n^X) \cap \text{cof}(\geq \mu^*)$ in (17) really means **all** limits of \tilde{D}_n^X of cofinality $\geq \mu^*$, not just those below $\sup(X \cap \omega_n)$. In particular, it includes the ordinal $\sup(X \cap \omega_n)$.⁸

Claim 23. *If $\eta \in \lim(\tilde{D}_n^X) \cap \text{cof}(\geq \mu^*)$ **and** $\text{cf}(\eta) < \omega_{n-1} \leq \eta$, then $o^K(\eta) \geq \text{cf}^V(\eta) \geq \mu^*$.*

Proof. Fix such an η . The assumptions of the claim guarantee that $\omega < \text{cf}(\eta) < |\eta|$ and that $\eta > \omega_2$; so by Theorem 18, to prove that $o^K(\eta) \geq \text{cf}^V(\eta)$ it suffices to prove that η is regular in K . Suppose for a contradiction that η is singular in K . In K , fix some continuous $\vec{\eta} = \langle \eta_i \mid i < \text{cf}^K(\eta) \rangle$ which is cofinal in η and such that $\eta_0 > \text{cf}^K(\eta)$. Then every member of

$$E := \{\eta_i \mid i \text{ is a limit ordinal}\}$$

is singular in K ,⁹ and moreover E is club in η . So in particular, almost every member of $\eta \cap \text{cof}(< \mu^*)$ is singular in K . This contradicts (17). \square

Claim 24. *The set of η which satisfy the assumptions of Claim 23 is stationary in ω_n .*

Proof. Note that $\mu_n \geq \mu^*$; we consider two cases, depending on whether this inequality is strict.

If $\mu_n = \mu^*$ then $\sup(X \cap \omega_n)$ is a μ^* -cofinal limit of \tilde{D}_n^X .¹⁰ Also, since $n \in I'$ then $\mu^* < \omega_{n-1}$, and so the cofinality of $\sup(X \cap \omega_n)$ is strictly less than ω_{n-1} . Finally, note that

$$\bigcup_{X \in T''} \{\sup(X \cap \omega_n)\}$$

is stationary in ω_n , because T'' is stationary.

⁸And $\sup(X \cap \omega_n)$ might be the *only* element of $\lim(\tilde{D}_n^X) \cap \text{cof}(\geq \mu^*)$, in the case that $\mu^* = \mu_n$.

⁹Because $\vec{\eta} \restriction i$ witnesses singularity of η_i .

¹⁰Possibly the only such limit of \tilde{D}_n^X ; i.e. in the case $\mu_n = \mu^*$, then at most nonstationarily many members of \tilde{D}_n^X are μ^* -cofinal.

If $\mu_n > \mu^*$, then $Q_n^X := \lim(\tilde{D}_n^X) \cap \text{cof}(\mu^*) \cap [\omega_{n-1}, \sup(X \cap \omega_n))$ is stationary (in fact μ^* -club) in $\sup(X \cap \omega_n)$ for all $X \in T''$. Also since $n \in I'$ then $\mu^* < \omega_{n-1}$. It follows that every $\eta \in Q_n^X$ satisfies the assumptions of Claim 23. Finally, note that because each Q_n^X is stationary in $\sup(X \cap \omega_n)$ and T'' is stationary, it follows that

$$\bigcup_{X \in T''} Q_n^X$$

is stationary in $\omega_n \cap \text{cof}(\mu^*)$, which completes the proof of the claim. \square

Thus Claims 23 and 24 imply that for any $n \in I'$, there are stationarily many $\eta < \omega_n$ such that $o^K(\eta) \geq \mu^*$. This completes the proof of Theorem 6.

4. STRONGER HYPOTHESES

In this section we shall prove Theorems 7,8 and 9. Let us start with Theorem 7. Let $l, m, \langle S_n : n > m + 1 \rangle, \langle n_k : k < \omega \rangle$ be as in its statement. As before we can find a stationary set T of $X \subseteq H_{\aleph_\omega}$ s.t $\sup(X \cap \aleph_n) \in S_n$ for all $n > m + 1$ and in the coiteration of K_X with K , which we can assume to be linear in this case, the K -side of the iteration drops and the K_X -side is trivial.

As before iteration indices are cofinal in β_ω^X and hence θ_X is a limit for all X . So we can fix an n^* such that whenever $\nu_i^X \geq \beta_{n^*}^X$ then there is no drop between i and θ_X . Also remember that whenever $k > n^*$ and $j < l$, then $\mu_{n_k} = \mu_{n_k+j}$. For such i that $\mu_i^X \geq \beta_{n^*}^X$ let us call the degree of elementarity of $\pi_{i,i+1}^X$ at that point m^* and let us refer to the - constant in i - cofinality of $\rho_{m^*}(N_i^X)$ as λ_X . Then there exist infinitely many $k > n^*$ s.t. $\mu_{n_k} \neq \lambda_X$.

An important difference is that we can no longer prove iteration indices to be cofinal in β_{n_k} even if $\mu_{n_k} \neq \lambda_X$. In fact, we will show that this is not the case! This is because our extenders might now have many generators.

Observation 25. *Let $k \geq m + 1$. Let $\alpha \in [\beta_{n_k}^X, \beta_{n_k+l}^X)$ be s.t. $K^X \models \exists \gamma : \alpha = \gamma^+$. Then $\text{cof}(\alpha) = \mu_{n_k}$.*

Proof. If $\alpha = \beta_{n_k+j}^X$ for some $j < l$ then this is by choice of our sequence. If not, then α is properly in between say $\beta_{n_k+j}^X$ and $\beta_{n_k+j+1}^X$ and thus by weak covering $\text{cof}(\sigma_X(\alpha)) = \aleph_{n_k+j}$. W.l.o.g X is closed under some function witnessing this. But this easily gives $\text{cof}(\alpha) = \text{cof}(\beta_{n_k+j}^X) = \mu_{n_k+j} = \mu_{n_k}$. \square

Lemma 26. *Let $k > n^*$ be s.t. $\mu_{n_k} \neq \lambda_X$. Then there exist an $i < \theta_X$ s.t. $\kappa_i^X < \beta_{n_k}^X \leq \nu_i^X$.*

Proof. Assume not. Because iteration indices are cofinal in β_ω^X there is some least i s.t $\nu_i^X > \beta_{n_k}^X$. By assumption we have $\kappa_i^X \geq \beta_{n_k}^X$. Then by coherence and the fact that there is no drop in between i and θ_X means that $((\beta_{n_k}^X)^+)^{K_X} = ((\beta_{n_k}^X)^+)^{M_i^X}$ is a regular cardinal of N_i^X . Furthermore, because by minimality of i all generators of the iteration up to this point are less than $\beta_{n_k}^X$, N_i^X is sound above $\beta_{n_k}^X$. Lastly, $((\beta_{n_k}^X)^+)^{K_X} \leq \rho_m(N_i^X)$ because there is no drop at i . So, Lemma 10 applies and gives us that $\text{cof}(((\beta_{n_k}^X)^+)^{K_X}) = \lambda_X$. On the other hand by Observation 25 $\text{cof}(((\beta_{n_k}^X)^+)^{K_X}) = \mu_{n_k}$. Contradiction! \square

Note here that by the same proof we have that $\text{cof}(((\kappa_i^X)^+)^{N_i^X}) = \lambda_X$ and thus it should be easy to see that $((\kappa_i^X)^+)^{N_i^X} < \beta_{n_k}^X$.

So for any $k > n^*$ s.t. $\mu_{n_k} \neq \lambda_X$ we can fix some i_k^X with $\kappa_{i_k^X}^X < \beta_{n_k}^X$ and $\nu_{i_k^X}^X \geq \beta_{n_k}^X$. To simplify our notation we shall henceforth refer to $\kappa_{i_k^X}^X$ as η_k^X , to $\nu_{i_k^X}^X$ as ζ_k^X , to the model $N_{i_k^X}^X$ as M_k^X and to the extender $E_{i_k^X}^X$ as F_k^X .

Lemma 27. *Let $k > n^*$ be s.t. $\mu_{n_k} \neq \lambda_X$. Then $((\zeta_k^X)^+)^{K_X} \geq \beta_{n_k+l}^X$.*

Proof. Consider $M^* := \text{Ult}(M_k^X, F_k^X)$. In M^* , $((\zeta_k^X)^+)^{N^*}$, which equals $((\zeta_k^X)^+)^{K_X}$ by coherence, is certainly regular and M^* is sound above ζ_k^X . Notice also that $\rho_{m^*}(M^*) \geq ((\zeta_k^X)^+)^{M^*}$. So, by Lemma 10 $\text{cof}(((\zeta_k^X)^+)^{M^*}) = \lambda_X$. On the other hand $((\zeta_k^X)^+)^{K_X}$ is a successor. card. in K_X ; if it were in the interval $[\beta_{n_k}^X, \beta_{n_k+l}^X)$, by Observation 25 it's cofinality would equal μ_{n_k} . So, we can conclude that $((\zeta_k^X)^+)^{K_X} \geq \beta_{n_k+l}^X$. \square

We can immediately conclude that $\text{o}^{K_X}(\eta_k^X) \geq \beta_{n_k+(l-1)}^X$ for every $k > n^*$ s.t. $\mu_{n_k} \neq \lambda_X$. By elementarity then $\text{o}^K(\sigma_X(\eta_k^X)) \geq \aleph_{n_k+(l-1)}$. Also, we know there exists infinitely many such k . So this concludes the proof of Theorem 7.

The proof gives a slightly stronger conclusion!

Fact 28. *Assume that $\xi \in (\sigma_X(\eta_k^X), \aleph_{n_k+(l-1)}) \cap \text{Card}^K$, then $\text{o}^K(\sigma_X(\eta_k^X)) \geq \xi$.*

We will need the above fact for the proof of Theorem 8:

Proof of Theorem 8. We will do the proof for cofinalities \aleph_1 and \aleph_2 , it is not hard to see that this case is representative. We just need to consider two sequences $\vec{S} := \langle S_{f(n)}^n : n \geq 8 \rangle$ and $\vec{T} := \langle S_{g(n)}^n : n \geq 4 \rangle$ where

$$f(n) = \begin{cases} 1 & n \bmod 8 = 0, 1, 2, 3 \\ 2 & n \bmod 8 = 4, 5, 6, 7 \end{cases}$$

$$g(n) = \begin{cases} 1 & n \bmod 4 = 0, 1 \\ 2 & n \bmod 4 = 2, 3 \end{cases}$$

Assume both \vec{S} and \vec{T} are mutually stationary. Using Theorem 7 we get a sequence $\langle \kappa_n : n < \omega \rangle$ and $\langle \lambda_n : n < \omega \rangle$ s.t. for all $n < \omega$ there exists $\kappa < \kappa_n$ and $\kappa' < \lambda_n$ with $\text{o}^K(\kappa) \geq \kappa_n^{+3}$ and $\text{o}^K(\kappa') \geq \lambda_n^+$.

As to the identity of the κ_n 's they are the \aleph_k 's with either k at least some number n^* and $k \bmod 8 = 0$ or $k \geq n^*$ and $k \bmod 8 = 4$. Similarly, the λ_n 's are the \aleph_k 's with either k at least some number n^* and $k \bmod 4 = 0$ or $k \geq n^*$ and $k \bmod 4 = 2$. Of course, we can assume the two n^* 's to be the same.

Our job is now to simply check all of the 4 possible combinations and see that there must be some overlap on the K -sequence. By symmetry it suffices to examine just two of those cases.

Take some k big enough with $k \bmod 8 = 0$. Assume there is some $\kappa < \aleph_k$ with $\text{o}^K(\kappa) \geq \aleph_{k+3}$. The first case we look at is that there is $\kappa' < \aleph_{k+2}$ with

$\text{o}^K(\kappa') \geq \aleph_{k+3}$. This then tells us that there must be some other $\kappa'' < \aleph_{k-2}$ with $\text{o}^K(\kappa'') \geq \aleph_{k-1}$. In our situation we have that $\kappa < \aleph_k$ is a regular cardinal in K thus by Fact 28 we have that $\text{o}^K(\kappa'') \geq \kappa$. If ν was the index of the order zero measure on κ then $K||\nu$ is a 0^\sharp type mouse.

The other case works similar. Assume now that $\kappa' < \aleph_k$ with $\text{o}^K(\kappa') \geq \aleph_{k+1}$ exists. Then we also have $\kappa'' < \aleph_{k+4}$ measurable in K . As before - but applying Fact 28 at κ instead - we actually have $\text{o}^K(\kappa) \geq \kappa''$ and thus 0^\sharp .

As mentioned before, the remaining two cases are dealt with by a symmetric argument. \square

Finally, the proof of theorem 9:

So let us fix $1 < m < \omega$ and a mutually stationary sequence $\vec{S} := \langle S_n : n > m \rangle$ s.t for all n , S_n concentrates on a fixed cofinality μ_n s.t. $\liminf_{m < n < \omega} \mu_n = \aleph_\omega$. It is easy to see that we can require all the μ_n to be uncountable.

We shall do the following proof in greater generality. The above hypothesis is almost certainly very strong, close to inconsistent even. We believe it should be possible to extract an inner model with a Woodin cardinal from the hypothesis. Considering that the consistency of the statement is unsure, it might not be a worthy pursuit to do so.

We assume for a contradiction:

- (a) K is a core model satisfying weak covering at all but finitely many cardinals;
- (b) if E is a total extender on the K' -sequence where $K' \trianglelefteq K$, κ is it's critical point and ν it's index, then ν is a successor cardinal in $\text{Ult}(K'; E)$ and $\text{cof}((\kappa^+)^{K'}) = \text{cof}(\nu)$;
- (c) there exists some $X \prec (H_\omega; \in, K \cap H_\omega, \dots)$ s.t. $\sup(X \cap \aleph_n) \in S_n$ for all $m < n < \omega$ and in the co-iteration of K and K_X which is not necessarily linear, K_X does not move and K drops along its main branch.

The above is satisfied if 0^\sharp does not exist as evidenced by the core model below 0^\sharp , except (b) which is not quite true, but we can make do by substituting $(\nu^+)^{\text{Ult}(K', E)}$ for ν , the former does have the right cofinality as shown in the proof of Lemma 27). We do not know if it is satisfied if there is no inner model with a Woodin cardinal.

So, let us write \mathcal{T}_X for the iteration tree on K and b_X for its main branch from assumption (c). Let θ_X be the length of \mathcal{T}_X , $\langle N_i^X, \kappa_i^X, \nu_i^X, m_i^X : i < \theta_X \rangle$ be the iteration's models, critical points, indices and degrees. As before we can show that b_X has limit type. So there is some n^* s.t whenever $\nu_i^X \geq \beta_{n^*}^X$ and $i \in b$ then $\text{cof}(\rho_{m_i^X}(N_i^X))$ is constant in i . Call this constant value λ_X . W.l.o.g. $\lambda_X < \mu_n$ for all $n > n^*$.

Observation 29. *Let $n > n^*$. Let $\alpha \in [\beta_n^X, \beta_{n+1}^X)$ be s.t. $K^X \models \exists \gamma : \alpha = \gamma^+$. Then $\text{cof}(\alpha) > \lambda_X$.*

Proof. If $\alpha = \beta_{n+1}^X$ then this is by choice of our sequence. If not, then by weak covering $\text{cof}(\sigma_X(\alpha)) = \aleph_n$. W.l.o.g X is closed under some function witnessing this. But this easily gives $\text{cof}(\alpha) = \text{cof}(\beta_n^X) = \mu_n > \lambda_X$. \square

We can now derive a contradiction finishing the proof of theorem 9:

Let $i + 1 \in b$ be s.t. $\nu_i^X \geq \beta_{n^*}^X$. On the one hand we have that ν_i^X is a successor cardinal of K_X . Thus by Observation 29 $\text{cof}(\nu_i^X) > \lambda_X$.

On the other hand by assumption (b) $\text{cof}(\nu_i^X) = \text{cof}((\kappa_i^X)^+)^{M_i^X}$; furthermore, $((\kappa_i^X)^+)^{M_i^X} = ((\kappa_i^X)^+)^{M_j^X}$ because of agreement between models in iteration trees and, crucially, the fact that there occur no more drops on b_X from this stage on. (Here j is the \mathcal{T}^X -predecessor of $i + 1$). Clearly though, $((\kappa_i^X)^+)^{M_j^X}$ is a regular cardinal of M_j^X , and that model is sound above κ_i^X . So, Lemma 10 applies and gives $\text{cof}((\kappa_i^X)^+)^{M_i^X} = \lambda_X$. Hence $\text{cof}(\nu_i^X) = \lambda_X$. Contradiction!

5. OPEN PROBLEMS

Question 30. *Is it possible to force, starting from a model with at most finitely many measurable cardinals, that the sequence $\langle S_0^2, S_1^3, S_1^4, S_0^5, S_1^6, S_1^7, \dots \rangle$ is mutually stationary?*

Question 31. *What is the upper bound for the existence of a mutually stationary sequence satisfying the hypothesis of Theorem 7?*

Question 32. *Is the hypothesis of Theorem 8 consistent relative to large cardinals?*

Question 33. *Does “ \aleph_ω is Jonsson” imply that there exists in a - possibly trivial - forcing extension $V[G]$ a mutually stationary sequence satisfying the hypothesis of Theorem 9 relative to $V[G]$?*

Question 34. *Is it possible to generate mutually stationary sequences not covered by Theorem 4, e.g. the sequence $\langle S_1^4, S_1^5, S_2^6, S_2^7, S_1^8, S_1^9, S^1 0_2, S^1 1_2, \dots \rangle$, using Theorem 5, i.e. is it possible to have $\text{cof}(\prod_{n \in A_0} \aleph_n) = \aleph_{\omega+1}$ and $\text{cof}(\prod_{n \in A_1} \aleph_n) = \aleph_{\omega+2}$ where $A_0 := \{n < \omega \mid n \bmod 4 = 0, 1\}$ and $A_1 := \{n < \omega \mid n \bmod 4 = 2, 3\}$ or vice versa?*

REFERENCES

- [1] Sean Cox, *Covering theorems for the core model, and an application to stationary set reflection*, Ann. Pure Appl. Logic **161** (2009), no. 1, 66–93, DOI 10.1016/j.apal.2009.06.001. MR2567927
- [2] James Cummings, Matthew Foreman, and Menachem Magidor, *Canonical structure in the universe of set theory. II*, Ann. Pure Appl. Logic **142** (2006), no. 1-3, 55–75, DOI 10.1016/j.apal.2005.11.007. MR2250537 (2007g:03063)
- [3] Matthew Foreman and Menachem Magidor, *Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\kappa(\lambda)$* , Acta Math. **186** (2001), no. 2, 271–300, DOI 10.1007/BF02401842. MR1846032 (2002g:03094)
- [4] Kecheng Liu, *Stationary Subsets of $[\aleph_\omega]^{<\omega}$* , Journal of Symbolic Logic **58** (1993), no. 4, 1201–1218, DOI 10.2307/2275139.
- [5] Ronald Jensen, *Forcing axioms compatible with CH*, Handwritten Notes. URL: <http://www.mathematik.hu-berlin.de/~raesch/org/jensen.html>.
- [6] Peter Koepke, *Forcing a mutual stationarity property in cofinality ω_1* , Proc. Amer. Math. Soc. **135** (2007), no. 5, 1523–1533 (electronic), DOI 10.1090/S0002-9939-06-08598-4. MR2276663 (2007k:03129)
- [7] Peter Koepke and Philip Welch, *On the strength of mutual stationarity*, Set theory, Trends Math., Birkhäuser, Basel, 2006, pp. 309–320.
- [8] Kecheng Liu and Saharon Shelah, *Cofinalities of elementary substructures of structures on \aleph_ω* , Israel J. Math. **99** (1997), 189–205, DOI 10.1007/BF02760682. MR1469093 (98m:03100)

- [9] Menachem Magidor, *Representing sets of ordinals as countable unions of sets in the core model*, Trans. Amer. Math. Soc. **317** (1990), no. 1, 91–126, DOI 10.2307/2001455. MR939805 (90d:03108)
- [10] P. Koepke and P. D. Welch, *Global square and mutual stationarity at the \aleph_n* , Ann. Pure Appl. Logic **162** (2011), no. 10, 787–806, DOI 10.1016/j.apal.2011.03.003. MR2803948 (2012f:03107)
- [11] Martin Zeman, *Inner models and large cardinals*, de Gruyter Series in Logic and its Applications, vol. 5, Walter de Gruyter & Co., Berlin, 2002. MR1876087 (2003a:03004)

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